Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

XI.

APPLICATIONS OF THE GRAPHICAL METHOD.

BY PROF. EDWARD C. PICKERING.

Read, May 12, 1874.

To establish any physical law, or the relation between two quantities so connected that a change in one produced a corresponding variation in the other, the following method is commonly adopted. A great many corresponding values of the two variables are determined with the greatest possible care, and corrected for all the errors to which they are known to be subject. Other errors known as accidental errors, however, still remain, whose minuteness depends on the excellence of the method of measurement and the care exercised. If now an equation can be found to satisfy all these values, its expression in common language will be the required law. Two methods are in common use to determine this equation. First, analytically, the form of the equation is assumed, and the values of the constants in it are found, by assuming certain of the observations to be correct; or, better, by the theory of probabilities employing all the observations, and computing from them the most probable values of these constants. The principal objection to this method is that it furnishes no means of discriminating between the accidental errors and the real variations from the law. In the second or Graphical Method, the two variables are taken as co-ordinates, and points are constructed corresponding to each observation. A curve is then drawn, coinciding with them as nearly as possible, and its equation determined by trial. The form of the curve shows very clearly the accidental errors or other causes of deviation; and the want of accuracy of this method, when the accidental errors are only small, may be completely overcome by the method of residual curves (Journal Franklin Instit., April, 1871).

There still remains, though in a less degree than in the Analytical Method, the difficulty of deciding whether a variation is due to errors, or to incorrect value of some of the constants taken. And it is to meet this difficulty that the following method is proposed.

With the exception perhaps of the circle, the straight line is the only curve of whose correctness every one is a judge; if, then, by any device we can so transform a curve that it shall become a straight line, a moment's inspection will show whether the agreement with observation is real.

Let us first take a special case, and then proceed to the more general discussion. A great many physical laws may be expressed by the equation $y = m\,x^n$, or that one quantity, y, is always proportional to some power of the other, x. For example, the variation of gravity, of the intensity of light, heat, and electric attraction, with the distance, may be stated $y = m\,x^{-2}$, or $n = -2$. For elastic forces, $n = 1$, or is proportional to the distance; for Mariotte's law, $n = -1$; for the deflection of a beam in terms of its length, $n = 3$, and so on. Sometimes n is a fractional number, or has a much larger value; thus Wertheim suggests that the laws of elasticity may be explained by assuming that the force of attraction of the particles varies at the 14th power of their distance apart. In the same way, Rankine adopts the exponents $n = -\frac{17}{16}$ and $n = -\frac{10}{9}$ for the variations of the pressure and volume of steam in the cylinder of an engine. Suppose now that we have a number of points constructed, and wish to see if they can be represented by any curve of the form $y = m\,x^n$. By drawing curves taking various values of n, as 1, 2, 3, $\frac{1}{2}$, &c., we may find one which will agree, but it will be difficult to be sure whether some other value will not give a more exact concordance. If, however, we take logarithms of both sides, and write $\log y = \log m + n \log n$, and calling $\log y = Y$, $\log n = X$, and $\log m = M$, construct a curve with Y and X as coordinates, we obtain $Y = M + n\,X$. If now the result is a straight line, or differs from a straight line only by the accidental errors, — that is, if there is no curvature to one side more than on the other, — we know that y varies as some power of x, and the value of n is readily determined from the tangent of the angle the line makes with the axis of X. In the same way m is obtained by finding the number whose logarithm is M, the ordinate of the point where the line meets the axis of Y. On the other hand, if the line is not straight, but curved, we may be sure that there is no value of n which will satisfy the observations, or that y is not proportional to *any* power of x. Let us next see how far this method may be generalized. In the first place, instead of x and y we may use any functions f and f' which include only x, y, and *known* constants; that is, which do not include m or n. For example, the equation $y^3 = m\,x^4 + n\,x^2$ may be written $\frac{y^3}{x^2} = m\,x^2 + n$, or calling $Y = \frac{y^3}{x^2}$ and $X = x^2$, we have an equation of a linear form, $Y = m X + n$. We may then in general write:—

$$f = m f' + n \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

And, if one equation can be reduced to this form, we can readily determine whether there are any values of m and n which will satisfy it. Again, if we have:—

$$f = m f'^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

we can reduce to a linear form by using $Y = \log f$ and $X = \log f'$. Another common case is:—

$$f = m\, n^{f'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Taking logarithms $\log f = \log m + f' \log n$, from which $\log m$ and $\log n$, and hence m and n, are readily found. The most important application of this formula is when two quantities are so connected that if one varies in arithmetical progression, the other will vary geometrically. This is the case for the variations of the barometer for various heights, for the conduction of heat, and the loss of potential of an insulated cable by leakage. In all these cases we have $y = m\, n^x$, where x varies arithmetically and y geometrically, and from which m and n may be determined as above.

Probably the best way of illustrating these principles is by a few examples, in which, however, figures would be required to show the results most clearly.

1. *Torsion Pendulum.* Four observations were made on the time of vibration of a torsion pendulum when its length was varied. The results are given in columns 1 and 2 of the adjoining table. The 3d and 4th columns give their logarithms. On constructing the points with these co-ordinates, they fall very nearly on a straight line, as is shown by column 5, which gives values of log t, computed by the for-

TABLE I.

Length.	Time.	Log l.	Log t.	T.
1201.	6.3	3.08	.80	.80
949.	5.7	2.98	.76	.75
706.	4.8	2.85	.68	.68
445.	3.8	2.68	.58	.60

mula $C = \frac{1}{2} \log l + .75$. The close agreement shows that $t = m \sqrt{l}$, or that the time is proportional to the square root of the length.

2. *Force of Magnetism.* The observations given in Table II. were made by Professor Mayer (Amer. Jour. Sci., Sept. 1870), to determine the effect of a coil on a galvanometer needle placed at different distances.

TABLE II.

D.	Log *D.*	Log *f.*	*C.*
4	.602	.897	.900
5	.699	.627	.630
6	.778	.413	.413
7	.845	.234	.230
8	.903	.072	.071

The first column gives the distance, the second its logarithm, and the third the logarithm of the force produced, or the tangent of the angle of deflection. To see if $f = m\, d^n$, a curve was constructed, with columns 2 and 3 as co-ordinates, and appeared to coincide very closely with the line $y = -2.76\, x + 2.545$. Column 4 gives the values of log f thus computed, which shows a close agreement with observation. The result found by Professor Mayer was $n = 2.7404$; but the last two figures should be omitted, as they alter the result by only about one or two hundredths as much as the accidental errors.

3. *Resistance of Air.* Another excellent example is found in the resistance of air to projectiles. Newton assumed that the resistance was proportional to the square of the velocity, or $R = m\, v^2$; but this result is not sustained experimentally. The agreement with the cube of the velocity is, in fact, more exact; but neither is the true law. A more careful examination shows that the law alters for velocities above and below that of sound, or about 1,100 feet per second; since above that velocity the air cannot flow in rapidly enough to fill the space behind the shot, but leaves a vacuum. To show this, a series of observations with the Bashforth chronograph were examined, and showed in a marked manner the change when $i = 1{,}100$. No part of the curve, however, for either spherical or elongated shot becomes a straight line; and therefore no power of the velocity will give the correct value of the resistance.

4. *Conic Sections.* It often happens, especially in astronomy, that

having a number of points we wish to see if they lie on any curve of the second degree. For instance, suppose the polar co-ordinates of the various points given, with the pole at the focus: then $r = \frac{m}{1 + n \cos v}$, in which we wish to see if any values of m and n satisfy all the conditions. The equation may be written $\frac{1}{r} = \frac{1}{m} + \frac{n}{m} \cos v$, which becomes linear if $\frac{1}{r} = Y$, and $\cos v = X$. In the same way, if referred to its centre, $\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1$, make $X = x^2$ and $Y = y^2$, when $\frac{1}{m^2}$ and $\frac{1}{n^2}$ are obtained.

5. *Periodic Functions.* In the study of periodic functions the equation $y = m \sin (n x + p)$ is assumed, in which m determines the maximum amplitude, n the period, and p the phase. If m is given $= a$, n and p may be found by writing $n x + p = \sin^{-1}\left(\frac{y}{a}\right)$, and using as co-ordinates x and $\sin^{-1}\left(\frac{y}{a}\right)$. If the period is given, or $n = b$, we have $y = m \sin b x \cos p + m \cos b x \sin p$, or, dividing by $\cos p$, $\frac{y}{\cos b x} = m \cos p \operatorname{tang} b x + m \sin p$, in which we may make $\frac{y}{\cos b x} = Y$, $\operatorname{tang} b x = X$, and thus determine $m' = m \cos p$ and $n' = m \sin p$. From these two equations we finally obtain $\operatorname{tang} p = \frac{n'}{m'}$ and $m = \sqrt{n'^2 + m'^2}$.

6. *Lissajous' Curves.* The wonderful variety of curves obtained by Lissajous, by mirrors attached to tuning-forks, may all be reduced to straight lines by this method. They may all be represented by the equations $x = \sin v$ and $y = \sin (m v + n)$, in which m represents the interval of the forks, and n the difference of phase. Eliminating v, we have $m \sin^{-1} x + n = \sin^{-1} y$, which at once takes the linear form when the co-ordinates $X = \sin^{-1} x$ and $Y = \sin^{-1} y$ are employed. To test this, a curve was drawn with an instrument devised by the writer (Journ. Frank. Inst., Jan. 1869), and forty-eight points on it measured, corresponding to variations of v of 30°. In the following table a portion only of the results are given. The first column gives various values of v, the second the measured value of the arc whose sine is y, or of $m v + n$. Constructing a curve with these co-ordinates, we obtain very nearly a straight line, with equation $y = \frac{34}{45} v + 4°$, from which we infer that the difference in phase of the two forks at 0° was 4°, and their interval a little more than a fourth (3 : 4), $\frac{34}{45} = .755$. The differences, as given in the fourth column, are very small, considering the roughness of the measurements.

TABLE III.

v.	$m\,v + n$.	$\frac{34}{45}\,v + 4^\circ$.	D.
0	0.	4.	—4.
180	139.	140.	—1.
360	276.	276.	0.
540	412.	412.	0.
720	544.	548.	—4.
900	685.	684.	+1.
1080	817.	820.	—3.

But it is needless to multiply these examples further. The method, in combination with that of residual curves, may be applied to almost any case where an empirical formula is to be deduced, and enables us to find the best values of two of the constants. A wide field seems to be open for it in astronomy; and it is to be regretted that heretofore astronomers should have neglected the graphical methods of discussing errors, as much as physicists have overlooked the more rigid analytical methods.